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AFFINE TRANSFORMATION THEORY OF THE CUSPIDAL CUBICS

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E.F.SIMONDS.

INTRODUCTION.

This problem was suggested by a re-reading of Halphen's famous thesis on Differential Invariants.†

The first part is concerned with the derivation of the affine invariants, and the examination of their properties. The absolute invariants are of orders 5 and 6 respectively; we have called them u and v .

The remainder of the paper deals with invariant families of cuspidal cubics. We find

(a) three 5-parameter families, $u = 25$, $u = -\frac{64}{5}$, $u = -\frac{1}{5}$;

(b) three special 6-parameter families, $P = 0, Q = 0, R = 0$;

(c) a single infinity of families $QR^2 = cP^3$.

The last is the general solution of Halphen's equation for all cuspidal cubics, represented as a first-order equation in u and v.

Finally, the areal coordinates of a point on the cubic with regard to the intrinsic triangle are expressed in terms of P , Q , and R .

Given the transformations of the affine group

$$X = ax + by + c, \qquad Y = dx + ey + f$$

we find that $Y'' = \Delta\,\theta^{-\frac{1}{3}}\,y''$, where $\Delta = ae - bd$, $\theta = a + by'$.

Writing $p = y''^{-\frac{1}{3}}$, $P = Y''^{-\frac{1}{3}}$, we see that

(I) $$P = \Delta^{-\frac{1}{3}}\,\theta\,p.$$

A short computation gives

(2) $$\frac{d}{dx}\left(p\,\frac{dp}{dx}\right) = \Delta^{\frac{2}{3}}\;\frac{d}{dX}\left(P\,\frac{dP}{dX}\right)$$

showing that $\frac{d}{dx}\left(p\,\frac{dp}{dx}\right)$ is a relative invariant subject to the magnifying factor $\Delta^{-\frac{2}{3}}$.

Now since $\frac{d}{dX} = \theta^{-1}\frac{d}{dx}$, we have, on combining with (I)

(3) $$P\,\frac{d}{dX} = \Delta^{-\frac{1}{3}}\,p\,\frac{d}{dx}$$

so that from any known invariant I with a constant magnifying factor we derive one of higher order, viz., $p\,\frac{dI}{dx}$.

Thus we have an infinite system of relative invariants

$$p, \quad \frac{d}{dx}\left(p\frac{d}{dx}\right), \quad p\frac{d}{dx}\left\{\frac{d}{dx}\left(p\frac{dp}{dx}\right)\right\}\cdots\cdots\cdots$$

all of which, with the exception of the first, are absolute invariants of the area-preserving subgroup for which $\Delta = 1$. The corresponding magnifying factors are

$$\Delta^{-\frac{1}{3}}\theta, \quad \Delta^{-\frac{2}{3}}, \quad \Delta^{-1}, \quad \Delta^{-\frac{4}{3}} \quad \cdots\cdots$$

Discarding numerical factors, we shall write some of the invariants ( ) as follows, denoting the order by a subscript;

$$\varphi_2^{-\frac{1}{3}} = y''^{-\frac{1}{3}} = J_2^{-\frac{1}{3}}$$

$$\varphi_4 = y''^{-\frac{8}{3}}J_4, \quad \varphi_5 = y''^{-4}J_5, \quad \varphi_6 = y''^{-\frac{16}{3}}J_6, \quad \varphi_7 = y''^{-\frac{20}{3}}J_7$$

where $$J_4 = 3y''y^{IV} - 5y'''^2$$

$$J_5 = 9y''^2y^V - 45y''y'''y^{IV} + 40y'''^3$$

$$J_6 = 9y''^3y^{VI} - 63y''^2y'''y^V - 45y''^2y^{IV^2} + 255y''y'''^2y^{IV} - 160y'''^4$$

$$J_7 = 27y''^4y^{VII} - 252y''^3y'''y^{VI} - 459y''^3y^{IV}y^V + 1395y''^2y'''^2y^V$$
$$+ 1980y''^2y'''y^{IV^2} - 5235y''y'''^3y^{IV} + 2560y'''^5.$$

J_2, J_4, J_5, J_6, J_7 are themselves relative invariants with the magnifying factors

$$\theta^{-3}\Delta \qquad \theta^{-8}\Delta^{2}, \quad \theta^{-12}\Delta^{3}, \quad \theta^{-16}\Delta^{4}, \quad \theta^{-20}\Delta^{5}$$

while φ_4, φ_5, φ_6, φ_7 have the factors $\Delta^{-\frac{2}{3}}$, Δ^{-1}, $\Delta^{-\frac{4}{3}}$, $\Delta^{-\frac{5}{3}}$ respectively. For future convenience we note here the following relations:-

$$(4) \qquad \frac{d\varphi_4}{dx} = \frac{1}{3}\varphi_2^{\frac{1}{3}}\varphi_5 \,, \quad \frac{d\varphi_5}{dx} = \varphi_2^{\frac{1}{3}}\varphi_6 \,, \quad \frac{d\varphi_6}{dx} = \frac{1}{3}\varphi_2^{\frac{1}{3}}\varphi_7 \,.$$

2. PROPERTIES OF THE INVARIANTS.

It will be observed that the J's are homogeneous in the y's, taking no account of the order of the derivatives. This is a property of any invariant of the affine group, for it must be true of all invariants of the transformation

$$X = x \,, \quad Y = ey.$$

We define the <u>weight</u> of the n^{th} derivative of y to be n + I. For example, each term of J_4 is of degree 2 and of weight 6. All affine invariants are isobaric,since this is a property of all invariants of the transformation

$$X = ax \,, \quad Y = y.$$

Since $Y^{(n)} = \Delta\theta^{-n-1}y^{(n)} +$ terms of lower order, it follows from the above that an affine invariant of degree m and weight n, when affected by the transformation

$$X = ax + by + c, \quad Y = dx + ey + f$$

is reproduced multiplied by $\Delta^m \theta^{-n}$.

When an invariant is expressed in terms of the J's it is clearly necessary that each term should have the same weight and the same degree in the y's. For example, in the invariant $J_4 J_6 - J_5^2$, J_4, J_5, and J_6 are of degrees 2, 3, and 4. and of weights 8, 12, and 16 respectively, so that the degree of each term in the y's is 6, and the weight 24. The invariant has therefore the magnifying factor $\Delta^6 \theta^{-24}$.

However, it is by no means necessary that an invariant should be homogeneous in the J's.

The above properties are characteristic of all projective invariants. However, the J's are not necessarily projective.

The reader will have noticed that amongst these quantities there is one,- J_2 -, which is exceptional. We shall now prove that if J_2^λ occurs as a factor in any term of an invariant expressed in terms of the J's, it is a factor of every term.

Suppose that the degree of the invariant in the y's is δ and the weight ω .This will be so for each term. Let $J_2^{\lambda} J_4^{x} J_5^{y} J_6^{z} \cdots$ be a term of the invariant. We have

$$\lambda + 2x + 3y + 4z + \cdots\cdots = \delta$$
$$3\lambda + 8x + 12y + 16z + \cdots\cdots = \omega$$

from which $\lambda = 4\delta - \omega$, and is the same for every term.

Now it is well known that under an r-parameter group there are but two independent absolute invariants of order not greater than r . Having regard to the magnifying factors of J_4 , J_5 , and J_6 , we see that the absolute invariants may be taken to be

$$u = J_5^2 J_4^{-3} \; ; \qquad v = J_6 J_4^{-2}$$

All other absolute invariants are functionally dependent on these two.

Let I be an invariant of degree δ and weight ω . Putting $y = x^{\lambda}$, I becomes $F(\lambda)\, x^{(\lambda+1)\delta-\omega}$. If then λ is a root of the equation $F(\lambda) = 0$, $y = x^{\lambda}$ is a particular solution of the invariant equation $I = 0$. The affine transforms of $y = x^{\lambda}$, viz.,

$$(5) \qquad dx + ey + f = (ax + by + c)^{\lambda}$$

are therefore solutions of $I = 0$.

The equation (5) involves five arbitrary parameters at most. We see therefore that every invariant differential equation of order $\not> 5$ has a general solution of the form (5).

For the curve $y = x^{\lambda}$ we find easily that

(6) $$u \equiv J_5^2 J_4^{-3} = - \frac{4(\lambda+1)}{(\lambda-2)(2\lambda-1)} = \alpha$$

It follows that the typical invariant differential equation of order 5 is

(7) $$J_5^2 = \alpha J_4^3$$

where α is a constant. The corresponding values of λ are given by

(8) $$(4+2\alpha)\lambda^2 + (8-5\alpha)\lambda + (4+2\alpha) = 0.$$

As the roots of this equation are reciprocals, they do not give essentially different solutions (λ) for the equation (5).

It will be noticed that in (5) $\lambda = -1$ gives the conics $J_5 = 0$; $\lambda = 2$ or $\frac{1}{2}$ gives the parabolas $J_4 = 0$. In the latter case the number of essential parameters in (5) is easily seen to be 4. $\lambda = 0$ or ∞ gives $\alpha = -2$; the transforms are those of $y = e^x$.

If λ is also a parameter, the differential equation of (5) is of the sixth order. We find that

$$v = J_6 J_4^{-2} = - \frac{2(\lambda+1)^2}{(\lambda-2)(2\lambda-1)} = \tfrac{1}{2}u.$$

Hence the ∞ 5-parameter families of curves whose differential equations are $J_5^2 = \alpha J_4^3$ form a six-parameter family whose equation is $2J_4 J_6 = J_5^2$. The general solution is

$$(dx + ey + f) = (ax + by + c)^{\lambda}$$

where a, b, c, d, e, f and λ are arbitrary.

This result could, of course, have been obtained by differentiating $J_5^2 - \alpha J_4^3 = 0$ and eliminating α. The operation is facilitated by the use of the ϕ's. Because the equation is isobaric, we can write it

$$(9) \qquad \phi_5^2 = \alpha\, \phi_4^3$$

by dividing by the appropriate power of y'' - in this case y''^8 - to reduce the weight to zero. We can now differentiate (9) by using the relations (4), and get

$$2\phi_6 = \alpha\, \phi_4^2$$

The elimination of α gives $2\phi_4 \phi_6 = \phi_5^2$, which can be replaced by

$$2J_4 \cdot J_6 = J_5^2.$$

However, the most general invariant differential equation of the sixth order is

$$F(u, v) = 0$$

where F is an arbitrary function.

We now come to an invariant of the seventh order.

3. HALPHEN'S PROJECTIVE INVARIANT OF ORDER 7.

This invariant, which we shall call H, is given by Halphen as the determinant

$$\begin{vmatrix} a_3 & a_4 & a_5 & a_6 & a_7 \\ a_2 & a_3 & a_4 & a_5 & a_6 \\ -a_2^2 & 0 & a_3^2 & 2a_3a_4 & 2a_3a_5 + a_4^2 \\ 0 & a_2^2 & 2a_2a_3 & 2a_2a_4 + a_3^2 & 2a_2a_5 + 3a_3a_4 \\ 0 & 0 & a_2^2 & 3a_2a_3 & 3a_3^2 + 3a_2a_4 \end{vmatrix}$$

where $a = \frac{1}{K!} y^{(K)}$.

As this is an invariant of the affine group, it is expressible in terms of J_4, J_5, J_6 and J_7; for a_2 is not a factor, therefore J_2 will not occur explicitly. From the 2-rowed determinant in the upper right-hand corner it is obvious that the terms J_5J_7 and J_6^2 will occur. These are of weight 32; and further consideration of the weights shows that H must be of the form

$$H = aJ_5J_7 + bJ_6^2 + cJ_4^4 + dJ_4J_5^2 + eJ_4^2J_6 .$$

Substitution in this identity of small numerical values gives us finally

$$H = \{28 \cdot 6^8 \cdot 20^2\}^{-1} \{2J_5J_7 - 7J_6^2 - J_4J_5^2\} .$$

Halphen has shown that the most general projective invariant equation of order seven is

$$H = k' J_5^{\frac{8}{3}} .$$

Discarding the numerical factor in H, we may write the equation

$$(10) \qquad 2J_5 J_7 - 7J_6^2 - J_4 J_5^2 = kJ_5^{\frac{8}{3}}$$

Equation (10) has a solution of the form $y = x^\lambda$, and is satisfied also by the ∞^7 projective transforms of $y = x^\lambda$,

$$(11) \qquad (dx + ey + f)(ax + by + c)^{\lambda-1} = (lx + my + n)^\lambda .$$

This solution involves seven essential parameters, except for special values of λ. $\lambda = 0$ or 1 would reduce the number to two, giving the straight lines; $\lambda = 2$ or $\frac{1}{2}$ would reduce it to five, giving the conics.

Since the number of essential parameters in (11) is at most 7, the general solution of an invariant equation of order greater than 7 is not of this type.

To find k in (10) we might substitute $y = x^\lambda$. The work is a little tedious; we can avoid it by the use of our auxiliary functions ϕ.

We recall that if $y = x^\lambda$, then $J_5^2 = \alpha J_4^3$, where

$$\alpha = - \frac{4(\lambda+1)^2}{(\lambda - 2)(2\lambda - 1)}$$

As previously explained, we have on dividing by an appropriate power of y'', $\phi_5^2 = \alpha\, \phi_4^3$, whence, on differentiating with the help of (4)

$$\phi_6 = \tfrac{1}{2}\alpha\phi_4^2$$

$$\phi_7 = \alpha\phi_4\phi_5 = \alpha^{\frac{3}{2}}\phi_4^{\frac{5}{2}}$$

(12) $$2\phi_5\phi_7 - 7\phi_6^2 - \phi_4\phi_5^2 = \tfrac{1}{4}\alpha(\alpha-4)\phi_4^4$$

Equation ( ) is not projective; but replacing ϕ_4^4 by its equivalent in terms of ϕ_5 we get

(13) $$2\phi_5\phi_7 - 7\phi_6^2 - \phi_4\phi_5 = \tfrac{1}{4}\alpha^{-\frac{1}{3}}(\alpha-4)\phi_5^{\frac{8}{3}}$$

which may be written

(14) $$2J_5J_7 - 7J_6^2 - J_4J_5^2 = kJ_5^{\frac{8}{3}}$$

where

$$k = \tfrac{1}{4}\alpha^{-\frac{1}{3}}(\alpha-4) = \frac{3(\lambda^2-\lambda+1)}{\{2(\lambda-2)(2\lambda-1)(\lambda+1)\}^{\frac{2}{3}}}$$

Equation (14) is, except for a numerical divisor and the cubing of both sides, Halphen's most general projective invariant equation of order 7.

The values $\lambda = 2, \frac{1}{2}$, or -1 give the conics; the roots of $\lambda^2-\lambda+1 = 0$ give, as shown by Halphen, the homographic transforms of the logarithmic spiral of $30°$. Since these values of λ make $\alpha = 4$, we see that the differential equation of the affine transforms of this spiral is $J_5^2 = 4J_4^3$.

Note that both roots of $\lambda^2-\lambda+1 = 0$ give the same solution, since they are reciprocals.

4. FIVE PARAMETER INVARIANT FAMILIES OF CUSPIDAL CUBICS.

The ∞^7 cuspidal cubics are the homographic transforms of $y = x^3$, viz.,

$$(dx + ey + f)(lx + my + n)^2 = (ax + by + c)^3$$

Their differential equation is obtained by putting $\lambda = 3$,

$$(15) \qquad 2J_5 J_7 - 7J_6^2 - J_4 J_5^2 = \frac{21}{4} \cdot 5^{-\frac{2}{3}} J_5^{\frac{8}{3}}$$

Let us now seek 5-parameter families invariant under the affine group. Each must satisfy a differential equation of the type $J_5^2 = \alpha J_4^3$, from which, as in ( )

$$(16) \qquad 2J_5 J_7 - 7J_6^2 - J_4 J_5^2 = \tfrac{1}{4}\alpha^{-\frac{1}{3}}(\alpha - 4) J_5^{\frac{8}{3}}.$$

Comparing (15) and (16) we get the equation for α;

$$\alpha - 21.\ 5^{\frac{2}{3}}\alpha^{\frac{1}{3}} - 4 = 0$$

whence $\alpha = 25$, $-\frac{64}{5}$, or $-\frac{1}{5}$. The corresponding values of λ are obtained from (6). They are respectively $\frac{3}{2}$, 3, or -2, or their reciprocals; hence

<u>There are but three five-parameter families</u> of cuspidal cubics invariant under the affine group, viz., the affine transforms of $y = x^{\frac{3}{2}}$, $y = x^3$, and $x^2 y = 1$.

These three types are indistinguishable in projective geometry; but they are not affine equivalents.

The reason for their occurrence is not far to seek. All cuspidal cubics have equations of the type $A^2 E = B^3$, where A, E, B are linear functions of x and y. $A = 0$ is the tangent at the cusp; $B = 0$ the line joining the cusp to the inflexion; and $E = 0$ the tangent at the inflexion. Now for the cubic $y = x^3$ the line at infinity is the tangent at the cusp, and the same will be true of its affine transforms. Similarly, in the transforms of $y = x^{\frac{3}{2}}$ the line at infinity is the tangent at the point of inflexion; while in the transforms of $x^2 y = 1$ the line at infinity joins the cusp to the inflexion.

We conclude then, that <u>the only 5-parameter families of cuspidal cubics invariant under the affine group are those for which one or other of the sides of the fundamental triangle is the line at infinity</u>.

5. THREE SPECIAL SIX-PARAMETER FAMILIES.

This leads us to examine the case of those cubics which have one vertex of the fundamental triangle on the line at infinity; an obvious invariant property.

There will be three such families. They should each involve six parameters, for only one degree of freedom is lost from the general case; two of the sides of the fundamental triangle are parallel.

By way of confirmation , it is a simple matter to verify that in each of the curves chosen to represent the respective families, the only affine transformation which fixes the curve is the identity.

The cubic $x^2 y = \frac{1}{6}(x+1)^3$ has its cusp at infinity and its inflexion at $(1,0)$. The tangent at the cusp is $x = 0$; that at the inflexion is $y = 0$. The third side of the fundamental triangle is $x + 1 = 0$. We find

$$J_4 = -x^{-10}(9x^2 + 24x + 20),$$

$$J_5 = -40x^{-15}, \qquad J_6 = 40x^{-20}(3x-1),$$

whence

$$u \equiv J_5^2 J^{-3} = -1600\,(9x^2 + 24x + 20)^{-3}$$

$$v \equiv J_6 J_4^{-2} = 40(3x-1)(9x + 24x + 20)^{-2}.$$

The elimination of x leads to

$$(17) \qquad Q \equiv v^2 + 5^{\frac{2}{3}} u^{\frac{2}{3}} v + u + \tfrac{21}{4} \cdot 5^{-\frac{2}{3}} u^{\frac{4}{3}} = 0.$$

This equation is satisfied by all the affine transforms of $x^2 y = \frac{1}{6}(x+1)^3$, for u, v are absolute invariants; it is therefore the differential equation of all cuspidal cubics having the cusp on the line at infinity.

We should expect (17) to be satisfied by all the cubics of the two 5-parameter families for which $A = 0$ or $B = 0$ is the line at infinity, for the cusp is at the point $A = B = 0$. Putting $v = \frac{1}{2}u$, the equation for u has the roots 0, $-\frac{64}{5}$, $-\frac{1}{5}$ and -25. $u = 0$ gives the conics; $u = -\frac{64}{5}$ and $u = -\frac{1}{5}$ give respectively the transforms of $y = x^{\frac{3}{2}}$ and $x^2 y = 1$ as expected. $u = -25$ is not a family of cubics.

Again, in the cubic $y^3 = x^2(x+1)$, the tangents at the cusp and the inflexion meet on the line at infinity. We find for this curve

$$u = -320^2(324x^2 + 288x - 16)^{-3}$$

$$v = 640(5 + 9x)(324x + 288x - 16)^{-2}$$

whence, on eliminating x,

$$(18) \qquad P \equiv v^2 - 5^{-\frac{1}{2}}u^{\frac{2}{3}}v + u - \frac{19}{4}\, 5^{-\frac{2}{3}}u^{\frac{4}{3}} = 0$$

This is the differential equation of all cuspidal cubics such that the tangents at the cusp and the inflexion meet at infinity. Putting $v = \frac{1}{2}u$, we find $u = 0$, $-\frac{64}{5}$, 25 and $\frac{1}{5}$. The second and third of these give two of the three invariant 5-parameter families. $u = \frac{1}{5}$ is not a family of cubics.

Finally, by using the curve $y^2(x+1) = x^3$ we get

$$(19) \qquad R \equiv v^2 - 4.5^{-\frac{1}{2}}u^{\frac{2}{3}}v + u + \frac{11}{4}\cdot 5^{-\frac{2}{3}}u^{\frac{4}{3}} = 0$$

as the differential equation of all cuspidal cubics having the inflexion at infinity. $v = \frac{1}{2}u$ gives $u = 0$, $-\frac{1}{5}$, 25, or $\frac{64}{5}$, the last not being a family of cubics.

We have found then, three independent invariant 6-parameter families $P = 0$, $Q = 0$, and $R = 0$. Each of these is characterised by the property that the corresponding vertex of the fundamental triangle is on the line at infinity.

7. HALPHEN'S EQUATION AS A DIFFERENTIAL EQUATION IN u,v.

Our affine form of Halphen's equation for all cuspidal cubics, viz.,

$$2J_5 J_7 - 7J_6^2 - J_4 J_5^2 = \frac{21}{4} \cdot 5^{-\frac{2}{3}} J_5^{\frac{8}{3}}$$

can be written

$$2u^{\frac{1}{2}}w - 7v^2 - u - \frac{21}{4} \cdot 5^{-\frac{2}{3}} u^{\frac{4}{3}} = 0\,, \tag{20}$$

where

$$u = J_5^2 J_4^{-3} = \phi_5^2 \phi_4^{-3}$$
$$v = J_6 J_4^{-2} = \phi_6 \phi_4^{-2}$$
$$w = J_7 J_4^{-\frac{5}{2}} = \phi_7 \phi_4^{-\frac{5}{2}}$$

We get from (4)

$$\frac{du}{d\phi_4} = \phi_4^{-1}(6v - 3u)$$

$$\frac{dv}{d\phi_4} = \phi_4^{-1}(u^{-\frac{1}{2}}w - 2v)$$

$$u^{-\frac{1}{2}}w = (6v - 3u)\frac{dv}{du} + 2v$$

Substituting for w in (20), we obtain the equation required, viz.,

$$(21) \qquad \xi \frac{dv}{du} + \eta = 0$$

where

$$(22) \qquad \begin{aligned} \xi &= 12uv - 6u^2 \\ \eta &= 4uv - 7v^2 - u - \tfrac{21}{4}\cdot 5^{-\frac{2}{3}} u^{\frac{4}{3}}. \end{aligned}$$

Such an equation was to be expected from the general theory of Lie.

8. GENERAL SOLUTION OF (21) IN TERMS OF P, Q ,R .

If $P = 0$ is a solution of (21), $\xi \frac{\partial P}{\partial u} - \eta \frac{\partial P}{\partial v} = 0$ when $P = 0$. We find that

$$\xi \frac{\partial P}{\partial u} - \eta \frac{\partial P}{\partial v} = \lambda_1 P$$

$$\xi \frac{\partial Q}{\partial u} - \eta \frac{\partial Q}{\partial v} = \lambda_2 Q$$

$$\xi \frac{\partial R}{\partial u} - \eta \frac{\partial R}{\partial v} = \lambda_3 R$$

where

$$\lambda_1 = 14v - 5^{-\frac{1}{3}}u^{\frac{2}{3}} - 8u$$
$$\lambda_2 = 14v + 5^{\frac{2}{3}}u^{\frac{2}{3}} - 8u$$
$$\lambda_3 = 14v - 4\cdot 5^{-\frac{1}{3}}u - 8u$$

showing that $P = 0$, $Q = 0$, $R = 0$ are solutions of (21) . Note that $3\lambda_1 = \lambda_2 + 2\lambda_3$. This suggests a solution of the type $QR^2 - cP^3 = 0$. We find, in fact, that

$$\left(\xi\frac{\partial}{\partial u} - \eta\frac{\partial}{\partial v}\right)(QR^2 - cP^3) = (\lambda_2 + 2\lambda_3)QR^2 - 3c\lambda_1 P^3.$$
$$3\lambda_1(QR^2 - cP^3)$$

As c is arbitrary, $QR^2 - cP^3 = 0$ is the general solution of (21).

It appears, then, that the 7-parameter totality of cuspidal cubics arrange themselves under the affine group into a single infinity of invariant 6-parameter families whose equations are $QR^2 - cP^3 = 0$, each family corresponding to a different value of c. These families have no significance for the general projective group; but c is an affine invariant.

9. DISCUSSION OF THE RESULT.

The cuspidal cubics $X^2(X+Y+1) = kY^3$ where k is a parameter all have same fundamental triangle .

$X = 0$ is the tangent at the cusp; $X+Y+1=0$ is the inflexional tangent; and $Y=0$ joins the cusp to the inflexion.

Now there is no affine transformation converting one of these curves into another. For the six available parameters are required to fix the fundamental triangle. Each vertex transforms into itself; this requires the identity.

In confirmation of this, let us consider the projective transformations which convert $X^2(X+Y+1)=kY^3$ into itself. It is readily verified that they form a one-parameter group

$$X = x\{(a^3-1)x+(a^3-a)y+a^3\}^{-1}$$
$$Y = ay\{(a^3-1)x-(a^3-a)y-a^3\}^{-1}$$

The affine transformations of this group would require $a^3-1=0$ and $a^3-a=0$ simultaneously. We get only $a=1$, which gives the identity.

Similarly the projective transformations converting $X^2(X+Y+1)=kY^3$ into $x^2(x+y+1)=hy^3$ are the one-parameter family - not, however, forming a group -

$$X = hx\left\{(ka^3-h)x+(ka^3-ha)y+ka^3\right\}^{-1}$$
$$Y = hay\left\{(ka^3-h)x+(ka^3-ha)y+ka^3\right\}^{-1}$$

This family contains no affine transformations, since there is no common solution to $ka^3-h=0$ and $ka^3-ha=0$.

It is clear, then, that the affine transforms of $X^2(X+Y+1)=kY^3$ arrange themselves into a single infinity of 6-parameter families, each corresponding to a different value of k. These must be the same families whose differential equations are $QR^2-cP^3=0$. It follows that c is a function of k. What is this function?

Obviously we must find P, Q, R for the curve $X^2(X+Y+1)=kY^3$. The direct calculation is a task not to be lightly undertaken. We can avoid it by transforming the equation by the projective transformation

(23) $$X=y^{-1}, \quad Y=xy^{-1}$$

which gives

(24) $$1+x+y=kx^3$$

The affine invariants J_4', J_5', and J_6' for (24) can be at once written down, since $y^{IV}=y^{V}=y^{VI}=0$; they are

$$J_4'=-5(6k)^2\ ;\quad J_5'=40(6k)^3\ ;\quad J_6'=-160(6k)^4$$

Now J_4 and J_6 are not projective invariants, so it is necessary to find the relations between them and their values J_4' and J_6' after transformation by ( )

We obtain easily

$$J_4 = y^{10}y'^{-8}(J_4' + 18y''^3y^{-1} - 9y'^2y''^2y^{-2} - 6y'y''y'''y^{-1})$$

$$J_5 = y^{15}y'^{-12}J_5'$$

$$J_6 = y^{20}y'^{-16}(J_6' + 3y'y''y^{-1}J_5')$$

We have from ( )

$$PJ_4^4 = J_6^2 - 5^{-\frac{1}{3}}J_5^{\frac{4}{3}}J_6 + J_4J_5^2 - \frac{19}{4}\cdot 5^{-\frac{2}{3}}J_5^{\frac{8}{3}}$$

and we find without undue labour that

(25) $$P = 9.5^{-\frac{2}{3}}u^{\frac{4}{3}}Y$$

and similarly

(26) $$Q = 27.5^{-\frac{2}{3}}u^{\frac{4}{3}}(X+Y+1)$$

(27) $$R = -\frac{27}{2}.\,5^{-\frac{2}{3}}u^{\frac{4}{3}}X$$

It follows that

$$QR^2P^{-3} = \frac{27}{4}X^2(X+Y+1)Y^{-3}$$

or, from the equation of the curve

$$QR^2 = \frac{27}{4}kP^3$$

Thus it turns out $c = \frac{27}{4}k$.

We now proceed to show the relation of P, Q, R with the areal coordinates of a point on the cubic.

IO. AREAL COORDINATES IN TERMS OF P, Q, R.

The areal coordinates p, q, r of the point (X,Y) on the cubic $X^2(X+Y+1)=kY^3$ with regard to the fundamental triangle are seen to be $-X$, $X+Y+1$ and $-Y$. They are affine invariants, for all areas are reproduced multiplied by Δ. From (25), (26) and (27) we may write them

$$(28)\qquad \begin{aligned} p &= -\frac{1}{9}\cdot 5^{\frac{2}{3}}u^{-\frac{4}{3}}P \\ q &= \frac{1}{27}\cdot 5^{\frac{2}{3}}u^{-\frac{4}{3}}Q \\ r &= \frac{2}{27}\cdot 5^{\frac{2}{3}}u^{-\frac{4}{3}}R \end{aligned}$$

Being invariants, they are also the areal coordinates of a point (x,y) on any of the transforms, with regard to the transformed fundamental triangle.

It is easy to verify by using the general expressions (18), (17), and (19) for P,Q,R that $p+q+r=1$.

We find from (28) that $qr^2p^{-3}=-k$. It follows that referred to areal coordinates p, q, r the intrinsic equation of <u>any</u> of the transforms of

$$X^2(X+Y+1)=kY^3$$

is $qr^2 + kp^3 = 0$; these coordinates are therefore the natural ones to use in the affine differential geometry of the cuspidal cubics.

REFERENCE.

G.H.HALPHEN, Oeuvres , Tome II ; pp. 197 - 253.